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# THE TWISTED SATAKE ISOMORPHISM AND CASSELMAN- SHALIKA FORMULA (Automorphic Representations and Related Topics)

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# THE TWISTED SATAKE ISOMORPHISM AND CASSELMAN-SHALIKA FORMULA

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ABSTRACT. For an arbitrary split adjoint group we identify the unramified Whittaker space with the space of skew-invariant functions on the lattice of coweights and deduce from it the Casselman-Shalika formula.

## 1. INTRODUCTION

The Casselman-Shalika formula is a beautiful formula relating values of special functions on a  $p$ -adic group to the values of finite dimensional complex representations of its dual group. Further, the formula is particularly useful in the theory of automorphic forms for studying  $L$ -functions.

In this note we provide a new approach to (and a new proof of) Casselman-Shalika formula for the value of spherical Whittaker functions.

To state our results we fix some notations. Let  $G$  be a split adjoint group over a local field  $F$ . We fix a Borel subgroup  $B$  with unipotent radical  $N$ , and consider its Levi decomposition  $B = NT$ , where  $T$  is the maximal split torus. Furthermore, we fix a maximal compact subgroup  $K$ .

Let  $\Psi$  be a non-degenerate complex character of  $N$ . For an irreducible representation  $\pi$  of  $G$  it is well known that  $\dim \operatorname{Hom}_G(\pi, \operatorname{ind}_N^G \Psi) \leq 1$  and in case it is non-zero we say that the representation  $\pi$  is *generic*. The Whittaker model of such a generic irreducible representation  $\pi$  of  $G$  is the image of an embedding

$$W : \pi \hookrightarrow \operatorname{Ind}_N^G \Psi.$$

Let now  $\pi$  be a generic irreducible representation. Recall that  $\pi$  is called *unramified* if  $\pi^K \neq \{0\}$  and that in this case  $\pi^K = \mathbb{C} \cdot v_0$ . Here  $v_0$  is a spherical vector. The explicit formula for the function  $W(v_0)$  was given in [CS] and is commonly called *the Casselman-Shalika formula*.

Recall that there is a bijection between irreducible unramified representations of  $G$  and the spectrum of the spherical Hecke algebra  $H_K = C_c(K \backslash G / K)$ . This commutative algebra admits the following description. Let  $\Lambda$  be the coweight lattice of  $G$ . Recall that  $\Lambda$  is canonically identified with  $T / (T \cap K)$ . The Weyl group  $W$  acts naturally on the lattice  $\Lambda$ . Denote by  $\mathbb{C}[\Lambda]^W$  the algebra of  $W$ -invariant elements in  $\mathbb{C}[\Lambda]$ .

The Satake isomorphism

$$S : H_K \simeq (\operatorname{ind}_{T \cap K}^T 1)^W = \mathbb{C}[\Lambda]^W$$

is defined by

$$S(f)(t) = \delta_B^{-1/2}(t) \int_N f(nt) dn$$

The main result of the paper is a description of the Whittaker spherical space  $(\text{Ind}_N^G \Psi)^K$  as a concrete  $H_K \simeq \mathbb{C}[\Lambda]^W$ -module. Namely, it is identified with the space  $\mathbb{C}[\Lambda]^{W,-}$  of functions on the lattice of coweights, that are skew-invariant under the action of the Weyl group  $W$ . More formally,

**Theorem 1.1.** *There is a canonical isomorphism*

$$j : (\text{ind}_N^G \Psi)^K \rightarrow \mathbb{C}[\Lambda]^{W,-},$$

*compatible with Satake isomorphism  $S : H_K \simeq \mathbb{C}[\Lambda]^W$ .*

From this result it easily follows that the twisted Satake map

$$S_\Psi : C_c(G/K) \rightarrow (\text{ind}_N^G \Psi)^K, \quad S_\Psi(f)(t) = \int_N f(nt) \overline{\Psi(n)} dn$$

sends the spectral basis of the spherical Hecke algebra  $H_K = C_c(K \backslash G / K)$  to the basis of characteristic functions of  $(\text{ind}_N^G \Psi)^K$ . In [FGKV], it is explained that this latter result is equivalent to the Casselman-Shalika formula in [CS] (see section 6 for a full account). Thus, we obtain a proof of the Casselman-Shalika formula that does not use the uniqueness of the Whittaker model.

Let us now quickly describe our proof of the main result. It was inspired by a new simple proof [S] by Savin of the Satake isomorphism of algebras

$$S : H_K \simeq (\text{ind}_{T \cap K}^T 1)^W = \mathbb{C}[\Lambda]^W.$$

Savin has observed that the Satake map  $S : C_c(G/K) \rightarrow (\text{ind}_N^G 1)^K$  restricted to

$$C_c(I \backslash G / K) = (\text{ind}_I^G 1)^K,$$

where  $I$  is an Iwahori subgroup, defines an explicit isomorphism

$$(\text{ind}_I^G 1)^K \simeq \text{ind}_{T \cap K}^T 1 = \mathbb{C}[\Lambda].$$

Restricting  $S_\Psi$  to  $(\text{ind}_I^G 1)^K$ , we prove there there exists an isomorphism

$$j : (\text{ind}_N^G \Psi)^K \rightarrow \mathbb{C}[\Lambda]^{W,-}$$

of  $H_K \simeq \mathbb{C}[\Lambda]^W$ -modules, making the following diagram

$$\begin{array}{ccc} (\text{ind}_I^G 1)^K & \xrightarrow{S_\Psi} & (\text{ind}_N^G \Psi)^K \\ \downarrow S & & \downarrow j \\ \mathbb{C}[\Lambda] & \xrightarrow{\text{alt}} & \mathbb{C}[\Lambda]^{W,-} \end{array}$$

commutative. Here the space  $\mathbb{C}[\Lambda]^{W,-}$  is a space of  $W$  skew-invariant elements of  $\mathbb{C}[\Lambda]$  and the alternating map  $alt$  are defined in the section 3.

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## 2. NOTATIONS

Let  $F$  be a local non-archimedean field and let  $q$  be a characteristic of its residue field. Let  $G$  be a split adjoint group defined over  $F$ . Denote by  $B$  a Borel subgroup of  $G$ , by  $N$  its unipotent radical, by  $\bar{N}$  the opposite unipotent radical, by  $T$  the maximal split torus and by  $W$  the Weyl group.

Denote by  $R$  the set of positive roots of  $G$  and by  $\Delta$  the set of simple roots. For each  $\alpha \in R$  let  $x_\alpha : F \rightarrow N$  denote the one parametric subgroup corresponding to the root  $\alpha$  and  $N_\alpha^k = \{x_\alpha(r) : |r| \leq q^{-k}\}$ .

Let  $\Psi$  be a non-degenerate complex character of  $N$  of conductor 1, i.e. for any  $\alpha \in \Delta$

$$\Psi|_{N_\alpha^0} \neq 1, \Psi|_{N_\alpha^1} = 1.$$

Let  $K$  be a maximal compact subgroup of  $G$ . Then  $T_K = T \cap K$  is a maximal compact subgroup of  $T$ . Choose an Iwahori subgroup  $I \subset K$  such that  $I \cap N = N_\alpha^1$  for all  $\alpha \in R$ . In particular  $\Psi|_{N \cap I} = 1$ , but  $\Psi|_{N_\alpha^0} \neq 1$  for any  $\alpha \in \Delta$ .

We fix a Haar measure on  $G$  normalized such that the measure of  $I$  is one.

The coweight lattice  $\Lambda$  of  $G$  is canonically identified with  $T/T_K$ . For any  $\lambda \in \Lambda$  denote by  $t_\lambda \in T$  its representative. The coweight  $\rho$  denotes the half of all the positive coroots. Since  $G$  is adjoint one has  $\rho \in \Lambda$ . We denote by  $\Lambda^+$  the set of dominant coweights.

Let  ${}^L G$  be the complex dual group of  $G$ . Then  $\Lambda$  is also identified with the lattice of weights of  ${}^L G$ . For a dominant weight  $\lambda$  we denote by  $V_\lambda$  the highest weight module of  ${}^L G$  and by  $wt(V_\lambda)$  the multiset of all the weights of this module.

## 3. FUNCTIONS ON LATTICES

Consider the algebra  $\mathbb{C}[\Lambda] = \text{Span}\{e^\nu : \nu \in \Lambda\}$ . The Weyl group  $W$  acts naturally on the lattice  $\Lambda$ . We denote by  $\mathbb{C}[\Lambda]^W$  the algebra of  $W$ -invariant elements in  $\mathbb{C}[\Lambda]$ . The character map defines an isomorphism of algebras

$$\text{Rep}({}^L G) \simeq \mathbb{C}[\Lambda]^W.$$

For an irreducible module  $V_\lambda$  denote

$$a_\lambda = \text{char}(V_\lambda) = \sum_{\nu \in \text{wt}(V_\lambda)} e^\nu.$$

The elements  $\{a_\lambda | \lambda \in \Lambda^+\}$  form a basis of  $\mathbb{C}[\Lambda]^W$ . The algebra  $\mathbb{C}[\Lambda]^W$  acts on the space  $\mathbb{C}[\Lambda]$  by multiplication.

The element  $f \in \mathbb{C}[\Lambda]$  is called *skew-invariant* if  $w(f) = (-1)^{l(w)} f$ , where  $l(w)$  is the length of the element  $w$ . Denote by  $\mathbb{C}[\Lambda]^{W,-}$  the space of  $W$  skew-invariant elements. The algebra  $\mathbb{C}[\Lambda]^W$  acts on  $\mathbb{C}[\Lambda]^{W,-}$  by multiplication. Note that the action is torsion free.

Define the alternating map

$$\text{alt} : \mathbb{C}[\Lambda] \rightarrow \mathbb{C}[\Lambda]^{W,-}, \quad \text{alt}(e^\mu) = \sum_{w \in W} (-1)^{l(w)} e^{w\mu} : \quad \mu \in \Lambda$$

It is a map of  $\mathbb{C}[\Lambda]^W$  modules. The elements

$$\{r_{\mu+\rho} = \sum_{w \in W} (-1)^{l(w)} e^{w(\mu+\rho)} : \quad \mu \in \Lambda^+\}$$

form a basis of  $\mathbb{C}[\Lambda]^{W,-}$ . Note that for any  $\lambda \in \Lambda^+$

$$\text{alt}(e^{\lambda+\rho}) = r_{\lambda+\rho} = r_\rho \cdot a_\lambda = \text{alt}(a_\lambda),$$

where the second equality is the Weyl character formula.

#### 4. HECKE ALGEBRAS

**4.1. The spherical Hecke algebra.** The spherical Hecke algebra  $H_K = C_c(K \backslash G / K)$  is the algebra of locally constant compactly supported bi- $K$  invariant functions with the multiplication given by convolution  $*$ . It has identity element  $1_K$  - the characteristic function of  $K$  divided by  $[K : I]$ .

Consider the Satake map

$$S : C_c(G/K) \rightarrow C(N \backslash G / K) = C(T/T_K)$$

defined by

$$S(f)(t) = \delta_B^{-1/2}(t) \int_N f(nt) dn.$$

The famous Satake theorem claims that the restriction of  $S$  to  $H_K$  defines an isomorphism of algebras  $S : H_K \simeq \mathbb{C}[\Lambda]^W$ . Denote by  $A_\lambda$  the element of  $H_K$  corresponding to  $a_\lambda$  under this map. Thus  $H_K = \text{Span}\{A_\lambda : \lambda \in \Lambda^+\}$ .

**4.2. The Iwahori-Hecke algebra.** The Iwahori-Hecke algebra  $H_I = C_c(I \backslash G / I)$  is the algebra of locally constant compactly supported bi- $I$  invariant functions with the multiplication given by convolution. Below we remind the list of properties of  $H_I$ , all can be found in [HKP].

- (1) The algebra  $H_I$  contains a commutative algebra  $A \simeq \mathbb{C}[\Lambda]$ .

$$A = \text{Span}\{\theta_\mu \mid \mu \in \Lambda\},$$

where

$$\theta_\mu = \begin{cases} \delta_B^{1/2} 1_{It_\mu I} & \mu \in \Lambda^+; \\ \theta_{\mu_1} * \theta_{\mu_2}^{-1} & \mu = \mu_1 - \mu_2, \quad \mu_1, \mu_2 \in \Lambda^+ \end{cases}.$$

The center  $Z_I$  of the algebra  $H_I$  is  $A^W \simeq \mathbb{C}[\Lambda]^W$ .

- (2) The finite dimensional Hecke algebra  $H_f = C(I \backslash K / I)$  is a subalgebra of  $H_I$ . The elements  $t_w = 1_{IwI}$ , where  $w \in W$  form a basis of  $H_f$ . Multiplication in  $H$  induces a vector space isomorphism

$$H_f \otimes_{\mathbb{C}} A \rightarrow H_I$$

In particular the elements  $t_w \theta_\mu$  where  $w \in W, \mu \in \Lambda$  form a basis of  $H_I$ .

- (3) The algebra  $H_K$  is embedded naturally in  $H_I$ . One has  $H_K = Z_I * 1_K$  and

$$A_\lambda = \left( \sum_{\nu \in \text{wt}(V_\lambda)} \theta_\nu \right) * 1_K$$

- (4) For the simple reflection  $s \in W$  corresponding to a simple root  $\alpha$  and a coweight  $\mu$  one has

$$t_s \theta_\mu = \theta_{s\mu} t_s + (1 - q) \frac{\theta_{s\mu} - \theta_\mu}{1 - \theta_{-\alpha}}.$$

In particular  $t_s$  commutes with  $\theta_{k\alpha} + \theta_{-k\alpha}$  for any  $k \geq 0$ . In addition  $t_s$  commutes with  $\theta_\mu$  whenever  $s\mu = \mu$ .

**4.3. The intermediate algebra.** Finally consider the space  $H_{I,K}$  defined by

$$H_K \subset H_{I,K} = H_I * 1_K = C_c(I \backslash G / K) \subset H_I.$$

It has a structure of right  $H_K$  module. The space  $H_{I,K}$  plays a crucial role in Savin's paper [S]. The Satake map restricted to it is the isomorphism of  $H_K \simeq \mathbb{C}[\Lambda]^W$  modules:

$$S : H_{I,K} \simeq \mathbb{C}[\Lambda], \quad S(\theta_\mu * 1_K) = e^\mu.$$

In particular, it is shown that the elements  $\{\theta_\mu^K = \theta_\mu * 1_K, \mu \in \Lambda\}$  form a basis of  $H_{I,K}$ .

## 5. THE WHITTAKER SPACE $(\text{ind}_N^G \Psi)^K$

Let  $\Psi$  be a non-degenerate character of conductor 1. Consider the space  $(\text{ind}_N^G \Psi)^K$  of complex valued functions on  $G$  that are  $(N, \Psi)$ -equivariant on the left, right  $K$ -invariant functions and are compactly supported modulo  $N$ .

The space  $(\text{ind}_N^G \Psi)^K$  has a structure of right  $H_K$  module by

$$(\phi * f)(x) = \int_G \phi(xy^{-1})f(y)dy, \quad \phi \in (\text{ind}_N^G \Psi)^K, f \in H_K.$$

Any function  $\phi$  on  $(\text{ind}_N^G \Psi)^K$  is determined by its values on  $t_\lambda : \lambda \in \Lambda$  and  $\phi(t_\lambda) = 0$  unless  $\lambda \in \Lambda^+ + \rho$ .

The space  $(\text{ind}_N^G \Psi)^K$  has a basis of characteristic functions  $\{\phi_\lambda : \lambda \in \Lambda^+ + \rho\}$  where

$$\phi_\lambda(ntk) = \begin{cases} \delta_B^{1/2}(t)\Psi(n) & t \in Nt_\lambda K \quad \lambda \in \Lambda^+ + \rho; \\ 0 & \text{otherwise} \end{cases}.$$

The main theorem of this paper is the description of  $(\text{ind}_N^G \Psi)^K$  as  $H_K$  module.

**Theorem 5.1.** *Let  $\Psi$  be a character of conductor 1. Then there is an isomorphism*

$$j : (\text{ind}_N^G \Psi)^K \simeq \mathbb{C}[\Lambda]^{W,-}$$

*compatible with  $H_K \simeq \mathbb{C}[\Lambda]^W$ .*

**5.1. The twisted Satake isomorphism.** For a fixed character  $\Psi$  of  $N$ , consider a twisted Satake map

$$S_\Psi : C_c(G/I) \rightarrow (\text{ind}_N^G \Psi)^I$$

defined by

$$S_\Psi(f)(t) = \int_N f(nt) \overline{\Psi(n)} dn.$$

**Corrolary 5.2.** *The restriction of  $S_\Psi$  to the right  $H_K$  submodule  $\theta_\rho^K * H_K$  defines an isomorphism*

$$S_\Psi : \theta_\rho^K * H_K \simeq (\text{ind}_N^G \Psi)^K$$

*such that  $S_\Psi(\theta_\rho^K * A_\lambda) = \phi_{\lambda+\rho}$ .*

*Proof.* By Weyl character formula  $r_{\lambda+\rho} = r_\rho \cdot a_\lambda$ . Hence

$$j(\phi_{\lambda+\rho}) = r_{\lambda+\rho} = r_\rho \cdot a_\lambda = j(\phi_\rho * A_\lambda),$$

and thus  $\phi_\rho * A_\lambda = \phi_{\lambda+\rho}$ .

Restricting  $S_\Psi$  to  $H_{I,K}$ , we obtain

$$S_\Psi(\theta_\rho^K * A_\lambda) = S_\Psi(\theta_\rho^K) * A_\lambda = \phi_\rho * A_\lambda = \phi_{\lambda+\rho}.$$

Since  $A_\lambda$  and  $\phi_{\lambda+\rho}$  are bases of  $H_K$  and  $(\text{ind}_N^G \Psi)^K$  respectively, the map  $S_\Psi$  is an isomorphism.  $\square$

To prove the theorem we shall need two lemmas. The first one ensures surjectivity of the map  $S_\Psi$  and the second one describes its kernel.

**Lemma 5.3.**  $S_\Psi(\theta_\mu^K) = \phi_\mu$  for all  $\mu \in \Lambda^+ + \rho$ . In particular the map

$$S_\Psi : H_{I,K} \rightarrow (\text{ind}_N^G \Psi)^K$$

is surjective.

*Proof.* It is enough to compute  $S_\Psi(\theta_\mu * 1_K)(t_\gamma)$  for  $\gamma \in \Lambda^+$ .

Since  $\mu$  is dominant one has

$$\theta_\mu^K = \delta_B^{1/2}(t_\mu) 1_{It_\mu K}$$

and hence

$$S_\Psi(\theta_\mu^K)(t_\gamma) = \delta_B^{1/2}(t_\gamma) \int_{N_{\gamma,\mu}} \overline{\Psi(n)} dn,$$

where

$$N_{\gamma,\mu} = \{n \in N : nt_\gamma \in It_\mu K\}.$$

The set

$$N_{\gamma,\mu} = \begin{cases} \emptyset & \gamma \neq \mu \\ N \cap K & \gamma = \mu \end{cases}$$

Indeed, since  $\mu \in \Lambda^+$  one has  $It_\mu K = (N \cap I)t_\mu K$ . One inclusion is obvious. For another inclusion use the Iwahori factorization

$$I = (I \cap N)T_K(I \cap \bar{N})$$

to represent any  $g \in It_\mu K$  as

$$g = na_0 \bar{n} t_\mu k = nt_\mu a_0 (t_\mu^{-1} \bar{n} t_\mu) k,$$

where  $n \in N \cap I, a_0 \in T_K, \bar{n} \in \bar{N}, k \in K$ . Since  $\mu$  is dominant one has  $(t_\mu^{-1} \bar{n} t_\mu) \in K$ . So  $g \in (N \cap I)t_\mu K$ . Hence  $N_{\gamma,\mu} = \emptyset$  unless  $\gamma = \mu$  and  $N_{\mu,\mu} = (N \cap I)t_\mu(N \cap K)t_\mu^{-1} = N \cap I$  since  $\mu \in \Lambda^+ + \rho$ . In particular  $\Psi|_{N_{\mu,\mu}} = 1$ . Hence

$$S_\Psi(\theta_\mu * 1_K) = \phi_\mu.$$

□

**Lemma 5.4.** Let  $\alpha \in \Delta$ ,  $s$  be a simple reflection corresponding to  $\alpha$  and  $\iota_\alpha = 1_I + t_s$  be the characteristic function of a parahoric subgroup  $I_\alpha$  corresponding to  $\alpha$ .

- (1)  $S_\Psi(\iota_\alpha) = 0$ .
- (2)  $S_\Psi(\theta_\mu^K + \theta_{s,\mu}^K) = 0$  for all  $\mu \in \Lambda$ .

*Proof.* 1)

$$S_\Psi(\iota_\alpha)(tw) = \int_N \iota_\alpha(ntw) \overline{\Psi(n)} dn = \int_{N \cap I_\alpha(tw)^{-1}} \overline{\Psi(n)} dn$$

The set  $N \cap I_\alpha(tw)^{-1}$  is empty unless  $w \in \{e, s\}$  and  $t \in T_K$ , in which case

$$S_\Psi(\iota_\alpha)(tw) = \int_{N \cap I_\alpha} \overline{\Psi(n)} dn = 0$$



since the integral contains an inner integral over  $N_\alpha^0$  on which  $\Psi$  is not trivial.

2) Let us represent any  $\mu = \mu' + k\alpha$  where  $(\mu', \alpha) = 0$ . Then  $s\mu = \mu' - k\alpha$ . In particular

$$\theta_\mu^K + \theta_{s\mu}^K = \theta_{\mu'}(\theta_{k\alpha} + \theta_{-k\alpha})\iota_\alpha 1_K$$

By the results in 4.2 the element  $i_\alpha$  commutes with  $\theta_{\mu'}(\theta_{k\alpha} + \theta_{-k\alpha})$  and hence the above equals

$$\iota_\alpha \theta_{\mu'}(\theta_{k\alpha} + \theta_{-k\alpha}) 1_K = \iota_\alpha (\theta_\mu^K + \theta_{s\mu}^K)$$

By part (1) it follows that  $S_\Psi(\theta_\mu^K + \theta_{s\mu}^K) = 0$ . □

*Proof.* of 5.1. We have shown that the map  $S_\Psi$  is surjective onto  $(\text{ind}_N^G \Psi)^K$  and

$$\text{Ker } S_\Psi = \text{Span}\{\theta_\mu - (-1)^{l(w)} \theta_{w\mu} \mid \mu \in \Lambda, w \in W\}.$$

Another words

$$(\text{ind}_N^G \Psi)^K \simeq H_{I,K} / \text{Ker } S_\Psi = \mathbb{C}[\Lambda]^{W,-}$$

as  $H_K \simeq \mathbb{C}[\Lambda]^W$ -modules. □

## 6. CASSELMAN-SHALIKA FORMULA

Let  $(\pi, G, V)$  be an irreducible smooth generic unramified representation and denote by  $\gamma \in {}^L T/W$  its Satake conjugacy class. Choose a spherical vector  $v_0$  and normalize the Whittaker functional  $W_\gamma \in \text{Hom}_G(\pi, \text{Ind}_N^G \bar{\Psi})$  such that  $W_\gamma(t_\rho v_0) = 1$ .

The Casselman-Shalika formula reads as follows:

**Theorem 6.1.**

$$W_\gamma(v_0)(t_{\lambda+\rho}) = \begin{cases} \delta_B^{1/2}(t_{\lambda+\rho}) \text{tr } V_\lambda(t_\gamma) & \lambda \in \Lambda^+ \\ 0 & \text{otherwise} \end{cases}$$

It is shown in [FGKV], that Theorem 5.2 that the formula (6.1) implies the Corollary 5.2 and it is mentioned that the two statements are equivalent. Let us now prove the other direction.

*Proof.* We deduce the formula 6.1 from 5.2. Let  $\pi$  be a generic unramified representation with the Satake parameter  $\gamma \in {}^L T$  and a spherical vector  $v_0$  and the Whittaker model  $W_\gamma : \pi_\gamma \rightarrow \text{Ind}_N^G \bar{\Psi}$  such that  $W_\gamma(v_0)(t_\rho) = 1$ . Define the map  $\chi_\gamma : H_K \rightarrow \mathbb{C}$  by

$$\pi(f)v_0 = \int_G f(g)\pi(g)v_0 dg = \chi_\gamma(f)v_0$$

and the map  $r_\gamma : \text{ind}_N^G \Psi \rightarrow \mathbb{C}$  by

$$r_\gamma(\phi) = \int_{N \backslash G} W_\gamma(v_0)(g)\phi(g) dg.$$

Then

$$\begin{aligned}
r_\gamma(S_\Psi(\theta_\rho^K * A_\lambda)) &= \int_{N \backslash G} \int_N (\theta_\rho^K * A_\lambda)(ng) \bar{\Psi}(n) W_\gamma(v_0)(g) dn dg = \\
&\int_G \int_G (\theta_\rho^K * A_\lambda)(g) W_\gamma(v_0)(g) dg = \\
&\int_G \int_G \theta_\rho^K(gx^{-1}) A_\lambda(x) W_\gamma(gx^{-1} \cdot x \cdot v_0)(1) dx dg = \chi_\gamma(A_\lambda) W_\gamma(v_0)(t_\rho).
\end{aligned}$$

Under the identification  $H_K \simeq \text{Rep}({}^L G)$  the homomorphism  $\chi_\gamma$  sends an irreducible representation  $V$  to  $\text{tr } V(\gamma)$ . In particular  $\chi_\gamma(A_\lambda) = \text{tr } V_\lambda(\gamma)$ .

$$\text{tr } V_\lambda(\gamma) = \chi_\gamma(A_\lambda) W(v_0)(t_\rho) =$$

$$r_\gamma(S_\Psi(\theta_\rho^K * A_\lambda)) = r_\gamma(\phi_{\lambda+\rho}) = \delta_B^{-1/2}(t_{\lambda+\rho}) W_\gamma(v_0)(t_{\lambda+\rho})$$

Hence

$$W_\gamma(v_0)(t_{\lambda+\rho}) = \delta_B^{1/2}(t_{\lambda+\rho}) \text{tr } V_\lambda(\gamma).$$

□

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